PhD Comprehensive Exam (Algebra)
Department of Mathematics
7 May 2026, 2:00 - 5:00 pm in MC 108

Instructions:

1. You have 3 hours to complete the exam.
2. Little partial credit will be given: aim for complete solutions.
3. You should attempt at least one question from each topic.
4. Justify all your answers.

Linear Algebra

1. Suppose A is an $n \times n$-matrix with coefficients in $\mathbb{Q}$ whose characteristic polynomial is irreducible over $\mathbb{Q}$. Let $v \in \mathbb{Q}^n \smallsetminus \{0\}$ be a non-zero vector and $V \leq \mathbb{Q}^n$ be the span of $B := \{A^i v : 0 \leq i < n\}$. Prove that $AV \subseteq V$ and that B is a basis of $\mathbb{Q}^n$.

2. Let $p \geq 2$ be prime. Let $V := \mathbb{F}_p^4$ with the standard $\mathbb{F}_p$-vector space structure and $W \leq V$ be a 2-dimensional $\mathbb{F}_p$-subspace. What is the number of invertible $\mathbb{F}_p$-linear transformations $A \colon V \to V$ satisfying $A(W) = W$?

Groups

3. Let G be the additive group of the field $\mathbb{Q}$. Show that every finitely generated subgroup of G is cyclic.

4. Suppose that G is a group of order pq^2, where $p < q$ are primes such that p does not divide $q^2 - 1$. Let P be a Sylow p-subgroup of G, and let Q be a Sylow q-subgroup of G. Prove that P and Q are both normal in G and deduce G is Abelian.

Rings and Modules

5. Prove that every nonzero prime ideal P in a PID R is a maximal ideal. Deduce that, if R is a commutative ring with identity such that $R[x]$ is a PID, then R is a field.

6. Suppose that M is an $\mathbb{R}[x]$-module such that the annihilator of M is the ideal of $\mathbb{R}[x]$ generated by $x^2(x^2+1)^2$. Suppose that M has dimension 8 when viewed as a real vector space. What is the number of possible isomorphism types of M?

Fields

7. Let E be a finite extension of $\mathbb{Q}$. Show the subgroup $\mu \subset E^\times$ of all roots of unity $\zeta \in E^\times$ is finite.

8. Suppose $\alpha \in \bar{\mathbb{Q}}$ has minimal polynomial $x^2 - x + 2 \in \mathbb{Q}[x]$. Find the minimal polynomial $\mu \in \mathbb{Q}[x]$ of α^2.