THE UNIVERSITY OF WESTERN ONTARIO
DEPARTMENT OF MATHEMATICS

Ph.D. Comprehensive Examination (Analysis)

May 14 2026, 2 - 5 PM

Instructions: This exam contains 8 problems. Solve as many problems as you can. Carefully justify your answers. Aim for complete solutions (there will be little or no credit for partial answers).

---

1. Find the closed, outward-oriented surface S in $\mathbb{R}^3$ for which the value of the integral

$$\iint_S \left(x - \frac{1}{3}x^3 - xy^2\right) dy\, dz + \left(y - \frac{1}{3}y^3\right) dz\, dx + \left(2z - z^3\right) dx\, dy$$

   is a maximum.

2. Consider the sequence of functions $f_n : \mathbb{R}^2 \to \mathbb{R}$ defined by:

$$f_n(x, y) = \frac{nxy}{1 + n^2(x^2 + y^2)^2}$$

   (a) Find the pointwise limit $f(x, y)$ of the sequence for all $(x, y) \in \mathbb{R}^2$.

   (b) Determine if the convergence is uniform on the closed unit disk $D_1 = \{(x, y) : x^2 + y^2 \leq 1\}$.

   (c) Determine if the convergence is uniform on the region $D_2 = \{(x, y) : x^2 + y^2 \geq 1\}$.

3. Consider the mapping $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by:

$$F(x, y) = (x + y + e^{xy}, x - y + e^{xy}).$$

   (a) Show that F is locally invertible in a neighbourhood of the origin $(0, 0)$.

   (b) Let G be the local inverse of F such that $G(F(0, 0)) = (0, 0)$. Compute the derivative matrix $DG(1, 1)$.

4. Let (X, d) be a complete metric space. Let $\mathcal{K}(X)$ denote the collection of all non-empty compact subsets of X. For any $A, B \in \mathcal{K}(X)$, the *Hausdorff distance* $d_H(A, B)$ is defined by:

$$d_H(A, B) = \max\left\{\sup_{a\in A} \operatorname{dist}(a, B), \sup_{b\in B} \operatorname{dist}(b, A)\right\}$$

   where $\operatorname{dist}(x, Y) = \inf_{y\in Y} d(x, y)$. Prove that $(\mathcal{K}(X), d_H)$ is a complete metric space.

5. Let $\Omega \subseteq \mathbb{C}$ be an open set, let $z_0 \in \Omega$, and let f be holomorphic on $\Omega \setminus \{z_0\}$. Suppose that

$$\lim_{z \to z_0} (z - z_0) f(z) = 0.$$

Show that f extends to a holomorphic function on all of Ω.

6. Compute the integral

$$I = \int_0^\infty \frac{x^2}{x^4 + 1}\, dx.$$

7. (a) Let f be holomorphic on the punctured disk $0 < |z - z_0| < r$. Classify the possible isolated singularities of f in terms of the principal part of its Laurent series, and give an example of each type (with proper justifications).

   (b) Define isolated singularities at infinity.

   (c) Give an example of a function with a non-isolated singularity at infinity.

8. Let $\lambda > 1$ be a real number. Show that the equation

$$\lambda - z - e^{-z} = 0$$

has exactly one solution in the right half-plane $\mathrm{Re}(z) > 0$.