THE UNIVERSITY OF WESTERN ONTARIO
London Ontario

Applied Mathematics Ph.D. Comprehensive Examination

27 May 2016
Part I: 9:30 am - 11:30 am

Instructions: The exam consists of Part I and Part II. Part I consists of mandatory problems and covers basic material. In Part I, 80% is required for a passing grade.

You may use a calculator, pen, and pencil. NO other aids are allowed. Your calculator must NOT be capable of wireless communication or capable of storing and displaying large text files.

PART I: Do ALL of the questions in the following four sections.

1. **Linear Algebra**

   (a) Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 4 & 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ a \end{pmatrix}$, and $\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.
   Find the row-echelon form of the augmented matrix $(A|\mathbf{b})$ for each $a \in \mathbb{R}$. Show how to use this echelon form to solve $A\mathbf{v} = \mathbf{b}$ where possible.

   (b) Suppose that B is a square matrix with eigenvalue λ.
   What is the defining property of λ? Show that this definition yields the condition $\det(B - \lambda I) = 0$.

   (c) Find the eigenvalues and corresponding eigenvectors of $\begin{pmatrix} c & 1 \\ 2 & c \end{pmatrix}$ for $c \in \mathbb{R}$.

2. **Calculus**

   (a) Find
   $$\lim_{x \to 0_+} x^{\sin^2 x}$$

   (b) Show that the equation
   $$x^3 - 15x + c = 0$$
   has at most one (real) root in the interval $[-2, 2]$. For what values of c this root exists?

   (c) Find
   $$\int \frac{\sin^3(\sqrt{x})}{\sqrt{x}}\, dx$$

   (d) Find the radius of convergence and the interval of convergence of the following series:
   $$\sum_{n=1}^{\infty} n!(2x-1)^n \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{1}{n!(2x-1)^n}$$

3. **Ordinary differential equations**

   (a) Give the general solution to $ty' + 5y - \ln t = 0\,,\ t > 0$.

   (b) Give the general solution to $y'' + 5y' + 6y = 84e^{4t}$.

   (c) Consider $y'' + xy' + 2y = 0$, and the point $x_0 = 0$. Find the recurrence relation for the coefficients of the power series solution.

   (d) For the following differential equation, make a substitution which results in a separable equation. Separate the variables. **Note: you should NOT solve the DE, just demonstrate that it is separable.**

$$\frac{dy}{dx} = \frac{-x + 3y}{x + 2y}$$

4. **Numerical Methods** Explicitly show how you obtain your numerical answers in the following.

   (a) i. Verify that $f(x) = 1 - \sin x$ and $g(x) = \cos^2 x/(1 + \sin x)$ are identical functions.

   ii. Which function should be used for computations when x is near $\pi/2$? Why?

   iii. Which function should be used for computations when x is near $3\pi/2$? Why?

   (b) Find a solution to $e^{-x^2} - x^2 = 0$ numerically to 6 digit accuracy. Explain why you believe you have 6 digit accuracy (Your answer should involve both backward and forward error descriptions).

   (c) Construct an interpolating polynomial through the points (-1,e^{-1}),(0,e^0),(1,e^1). Estimate the maximum error for this polynomial in approximating e^x over the interval $[-1, 1]$.

   (d) Consider the IVP $y' = e^t y$, on $0 \le t \le 1$, with $y(0) = 1$. Apply any *implicit* numerical method with a step of $h = 1/3$ to solve this problem.