THE UNIVERSITY OF WESTERN ONTARIO
London Ontario

Applied Mathematics Ph.D. Comprehensive Examination

Wednesday June 13, 2018
9:00 am - 12:00

The exam consists of two parts. Part I contains mandatory problems and covers basic material, while Part II covers slightly more advanced materials at graduate course level.

This is **Part I**, for which **80% is required to pass**. You may use a calculator, pen, and pencil. **NO other aids** are allowed. Your calculator **must NOT** be capable of wireless communication or capable of storing and displaying large text files.

PART I. Do all of the following questions.

Calculus:

C1. Evaluat the limit

$$\lim_{x\to 0} \frac{1}{x^6} \int_{x^3}^{x^2} \sin(t^2)\, dt.$$

C2. Consider the power series

$$\sum_{n=1}^{\infty} \frac{(x+2)^n}{(n+1)^p}.$$

where the parameter $p \in (0,1)$.

(a) Find the radius of convergence.

(b) Find the interval of convergence.

(c) Find the interval of absolute convergence.

C3. Prove that the equation $\frac{1}{2}e^x = \cos x$ has a **unique positive** solution.

C4. Find the conditions on the parameters a, b and c for which the function $f(x,y) = ax^2 + 2bxy + cy^2 + 2x + 4y + 6$ (i) has a minimum; (ii) has a maximum; and (iii) has no extrema.

C5. Evaluate the $\iint_\Omega (3x + 4y^2) dA$ where Ω is the region in the upper half of the $x - y$ plane bounded by the curves $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

C6. Evaluate the integral $\int x^2 e^{-x} dx$.

Linear Algebra:

L1. Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$.

(i) Find all eigenvaluse and the corresponding eigenvectors of A.

(ii) Determine whether or not A is diagonalizable, that is whether or not there exists an invertible matrix S such that $S^{-1}AS$ is an diagonal matrix. If yes, find S; if not, explain why.

(iii) Repeat (i) and (ii) for $B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$.

L2. Let $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$. Find conditions for a vector $u = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ to be in the span of v_1 and v_2, and under such condition(s), find the expression u in terms of v_1 and v_2.

L3. Express the function $f(x, y) = ax^2 + 2bxy + cy^2 + 2x + 4y + 6$ in [C4] by matrix forms, and use your knowledge on symmetric matrices (e.g. positive and negative definiteness) to explore the three questions (i), (ii) and (iii) in [C4].

Ordinary Differential Equations:

O1. Solve the equation $u'(t) + 2u(t) + 3t = 0$ with the initial condition $u(0) = 4$.

O2. Solve the logistic equation $u'(t) = ru(t)[1 - u(t)/K]$ with initial condition $u(0) = u_0$.

O3. Find the general solutions of the second order linear ODEs $u''(t) + 4u(t) = 3\cos t$ and $u''(t) + u(t) = 3\cos(2t)$. Comment/explain the difference of the solutions.

Numerical Methods:

Explicitly show how you obtain your numerical answers in the following.

N1. Write out the Taylor series for $\log x$ to order n about an arbitrary point $x_0 > 0$, *including the form of the remainder*. Using the expression for the remainder, and given that you pick a point within the radius of convergence of the series, derive a condition to determine a number of terms of the series that will guarantee an error less than some given tolerance tol. You do not need to solve for the number of terms, just find a condition that you could test after computing n terms to see if the error is less than tol.

N2. Numerically evaluate $\int_0^1 \exp(-x^2)dx$ using any reasonable means and estimate the error in your evaluation.

N3. (i) Most computer languages provide an $\exp(x)$ function and an $\text{expm1}(x)$ function (the later to evaluate $e^x - 1$). Why is the second one not redundant and for which arguments would it be most useful (explain why).

(ii) Most computer languages also provide a $\log(x)$ function and a $\text{log1p}(x)$ function (the later to evaluate $\log(1 + x)$). Why is the second one not redundant and for which arguments would it be most useful (explain why).

N4. Consider the equation

$$x \arctan x = 1.$$

Numerically solve for a value x (there is more than one solution but you only need to find one of them). Try to ensure that the forward error is less than 10^{-6} and explain why you believe you have achieved this accuracy in your solution. Evaluate the backwards error in your solution.