Ph.D. Comprehensive Examination — Part I

Tuesday June 8, 2021, 9:00-am-12:00 noon (Toronto Time)

The comprehensive exam consists of two parts. Part I contains mandatory problems and covers basic material, while Part II covers slightly more advanced materials at graduate course level.

This is **Part I**, for which a minimum of **80% is required to pass**. You may use a calculator, but your calculator **must NOT** be capable of wireless communication or capable of storing and displaying large text files. **NO other aids** are allowed.

Do **all** of the following questions.

Calculus:

C-1 Let $\alpha,\ \beta$ and γ be three positive real numbers satisfying $\alpha < \beta < \gamma$. Let $f(x) = (e^x)^\alpha\,,\ \ g(x) = x^\beta$ and $h(x) = (\ln x)^\gamma$ for $x \in (0, \infty)$. Order these three functions in terms of the rates of approaching ∞ as $x \to \infty$, starting from the one that grows slowest. You have to justify your ordering (show your work/analysis for your ordering).

C-2 Assume that $b(t)$ is Riemann integrable (not necessary differentiable) on $[-r, \infty)$ and $a, r > 0$ are positive real numbers. Let $u(t) = \int_0^r e^{-a\theta} b(t-\theta)\, d\theta$. Find a linear ordinary differential equation satisfied by $u(t)$.

C-3 Evaluate the iterated integral $\int_0^\infty \int_{x^2}^\infty x e^{-y^2}\, dy\, dx$.

C-4 Consider the power series $\sum_{n=1}^\infty \frac{[-(x-1)]^{n-1}}{2^{3n}}$.

(a) Find the interval of convergence.
(b) Find a formula for the sum function of this power series in the convergence interval.

C-5 Green's Theorem and Divergent Theorem in the plane.

(a) State the Green's Theorem for $\vec{F}(x, y) = (P(x, y), Q(x, y))$ in the plane in terms of the tangent along a closed curve C (including all conditions and the conclusion).
(b) State the Divergence Theorem in the plane for $\vec{F}(x, y)$ in the domain bounded by C in (a).
(c) Using the Green's Theorem and the relation between the tangent and the normal on the curve C to prove the Divergence Theorem.
(d) In the 1-dimension case, which theorem in calculus of single variables corresponds to the Divergence Theorem? Explain why you think so.

Linear Algebra:

L-1 Let $A = (a_{ij})_{n\times n}$ be a real and symmetric matrix.

(a) State as many criteria as you can for A to be positive definite.

(b) Determine whether or not the following matrix is positive definite: $A = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 0 & -3 \\ 1 & -3 & -1 \end{bmatrix}$.

L-2 Let W be the subspace in $\mathbb{R}^3$ spanned by $v_1 = (1, 2, 3)^T$.

(a) Find the orthogonal complement W^T of W with respect to the standard inner product.
(b) Geometrically interpret W^T.
(c) Find an orthonormal base for W^T.
(d) Find the orthogonal decomposition of the vector $v = (1, 1, 1)$ with respect to W and W^T.
(e) Find the Euclidean distance of $v = (1, 1, 1)$ to W^T.

L-3 Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 6 & 3 & 1 \end{bmatrix}$. Determine whether or not the vector $v = (1, -10, 5)$ is in the eigenvector space corresponding to the eigenvalue 1.

Differential Equations:

D-1 Use an appropriate substitute to find the general solution to the ODE $x^2 \frac{dy}{dx} = x^2 + xy - y^2$.

D-2 When considering heat equation $u_t(t, x) = ku_{xx},\ \ t > 0$ in the 1-D bounded domain (interval) $x \in [0, L]$ with homogeneous Neumann boundary condition $u_x(t, 0) = 0 = u_x(t, L)$, the method of separation of variables is typically applied, leading to the following ODE problem:

$$\begin{cases} \phi''(x) = -\mu\phi(x), x \in (0, L); \\ \phi'(0) = 0 = \phi'(L). \end{cases} \tag{1}$$

Find the conditions (show your work) on the parameter μ such that this ODE problem has nontrivial ($\phi \not\equiv 0$) solutions.

D-3 Solve the following linear ODE system with given initial values:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 5t \\ e^{2t} \end{pmatrix}, \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Numerical Methods:

N-1 On Newton's Method.

(a) Derive Newton's method for solving an algebraic equation $f(x) = 0$.

(b) For the particular case $f(x) = 2x^6 - 13x^4 + 24x^2 - 9$, apply Newton's method starting with $x_0 = 0.5$. Conduct 2 steps of the method, and obtain x_1 and x_2.

(c) How accurate is your estimate for the root?

(d) Repeat the method, now starting with $x_0 = 2.0$. How accurate is your estimate of the root?

N-2 Let A be an invertible $n \times n$ matrix and let b, Δb be $n \times 1$ matrices. Let x, Δx be unknowns.

(a) Given the equations

$$Ax = b\ , \tag{1}$$

$$A(x + \Delta x) = b + \Delta b\ , \tag{2}$$

prove that when b changes by an amount Δb, then the change in x, namely Δx, satisfies

$$\frac{\|\Delta x\|}{\|x\|} \leq \kappa(A)\, \frac{\|\Delta b\|}{\|b\|}\,, \tag{3}$$

where $\kappa(A) = \|A\|\, \|A^{-1}\|$.

(b) What name is given to the quantity $\kappa(A)$? Describe *briefly* (2 lines) the interpretation of equation (3) when κ is small, and when κ is large.

N-3 Consider the function

$$f(x) = \frac{1}{2+x}\ .$$

(a) Compute the first three terms of the Taylor series for $f(x)$ around $x = 0$.

(b) Compute the remainder term $R_2(\xi)$.

(c) For the case $x = 1$, obtain ξ explicitly.

(d) Use the series to approximate the integral

$$\int_0^1 \frac{dx}{2+x}\,,$$

and verify that the remainder term can be used to bound the error.