*The* UNIVERSITY *of* WESTERN ONTARIO

**Applied Mathematics Ph.D. Comprehensive Examination: Part 1**
**Date and time: 6 June 2022, 9:00am-12:00 Noon.**

**Answer all questions. A calculator is allowed, but no other aids.**

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# Calculus

1. (a) Write down or derive the Taylor series for $\sin x$ around $x = 0$.
   (b) Write down or derive the Taylor series for $\cos x$ around $x = 0$.
   (c) Hence, or otherwise, compute the limit

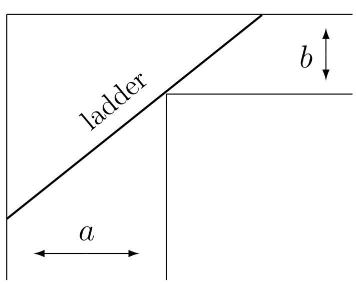
$$\lim_{x \to 0} \frac{\sin(x^2) - x^2 + x^6/6}{x^8(\cos x - 1)} \ .$$

**Answer:** (a) $\sin x = \sum(-1)^n x^{2n+1}/(2n+1)!$ (b) $\cos x = \sum(-1)^n x^{2n}/(2n)!$
(c) $\sin(x^2) - x^2 + x^6/6 = x^{10}/120 + O(x^{12})$ and $x^8(\cos x - 1) = -x^{10}/2 + O(x^{12})$. Limit$=-1/60$

2. Calculate the area between the curves $y_1 = x^3 + 3x^2 - x + 2$ and $y_2 = x^2 + 2x + 2$.

**Answer:** Curves cross at $x = -3, 0, 1$. $\int_{-3}^{0}(y_1 - y_2)\ dx + \int_0^1 (y_2 - y_1)\ dx = 45/4 + 7/12 = 71/6$.

3. A ladder must be carried around a corner joining two corridors. The corridors have widths $a$ and $b$. See the diagram below. What is the longest ladder that can be carried around the corner? You may approximate the ladder by a line, as shown.




**Answer:** (1) $L = \sqrt{a^2 + x^2} + \sqrt{b^2 + y^2}$ and $x/a = b/y$, so $L = (1 + b/x)\sqrt{a^2 + x^2}$. $L' = 0$ for $x = (a^2 b)^{1/3}$. Then $L = (a^{2/3} + b^{2/3})^{3/2}$.
(2) $L = a \sec\theta + b \csc\theta$, and $\tan^3\theta = b/a$.

4. Evaluate the integral

$$\int_0^\infty \int_{x^2}^\infty x e^{-y^2}\ dy\ dx \ .$$

**Answer:** Reverse limits: $\int_0^\infty x e^{-y^2}\ dydx = \int_0^\infty \int_0^{\sqrt{y}} x\ dx\ e^{-y^2}\ dy = \int_0^\infty (y/2) e^{-y^2} dy = 1/4$.

# Linear Algebra

5. Consider the matrix

$$Q = \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{pmatrix}$$

(a) Prove that this is an orthogonal matrix (also called orthonormal).

(b) Explain and prove why an orthogonal matrix is said to represent a rotation.

**Answer:** $Q^TQ = I$. For any vector $v$, $\|Qv\| = (Qv)^T Qv = v^T Q^T Qv = \|v\|$ so length is unchanged.

6. (a) Calculate the determinant of the matrix

$$A = \begin{pmatrix} 0 & 3 & 3 & 1 \\ 1 & 1 & 0 & 2 \\ 0 & 2 & 3 & 0 \\ 3 & -1 & 0 & 2 \end{pmatrix} .$$

(b) Given the results of the above calculation, describe the possible solutions for $x$ of a system of equations $Ax = b$, where $b$ is a random $4 \times 1$ vector.

**Answer:** (a) $\det A = 0$ (b) Either family of solutions or no solution depending on $b$.
Specifically solutions for $\langle p, q, r, 3q - 4p + 4r \rangle$.
The answer "no solution" is wrong.

7. Let $A$ be the matrix

$$A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix} .$$

(a) Calculate the eigenvectors of $A$.

(b) Diagonalize $A$.

(c) Hence or otherwise calculate $A^{10}$.

**Answer:** $\det(A - \lambda I) = (2 - \lambda)\det\begin{pmatrix} 0 - \lambda & -2 \\ 1 & 3 - \lambda \end{pmatrix} = (\lambda - 1)(\lambda - 2)^2$. $P = \begin{pmatrix} -2 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$
$P^{-1} = \begin{pmatrix} -1 & 0 & -1 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix}$. Then $A^{10} = P\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^{10} & 0 \\ 0 & 0 & 2^{10} \end{pmatrix} P^{-1} = \begin{pmatrix} -1022 & 0 & -2046 \\ 1023 & 1024 & 1023 \\ 1023 & 0 & 2047 \end{pmatrix}$

# Ordinary Differential Equations

8. By making an appropriate transformation, find the general solution of the ODE

$$x^2 \frac{dy}{dx} = x^2 + xy + y^2 .$$

**Answer:** $y = xu$, so $(xu)' = xu' + u$ and $xu' + u = 1 + u + u^2$. $xu' = 1 + u^2$. then $u'/(1+u^2) = 1/x$. So $\arctan u = \ln x + C$. $u = \tan(\ln x + C) = \tan(\ln(Kx))$, and $y = xu$.

9. Solve the initial value problem

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} - 2x = 0 \ ,$$
$$x(0) = 1 \ , \quad \dot{x}(0) = -2 \ .$$

What are the implications of your solution for solving this problem numerically? (only a brief answer is required – 2 lines maximum).

**Answer:** $x(t) = e^{-2t}$. The other solution grows exponentially and will magnify numerical errors.

10. Solve the boundary-value problem

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 6e^x \ ,$$
$$y(0) = 0 \ ,$$
$$y(\ln 2) = 3 \ .$$

**Answer:** Easiest: integrate equation: $\frac{dy}{dx} + 2y = 6e^x + c$. $y(x) = 2e^x - (4/3)e^{-2x} - 2/3$.

11. Consider the equation

$$y' = y + 2y^2 \ ,$$

with initial condition $y(0) = 1$. By differentiating the equation, or otherwise, obtain the Taylor series for the solution $y(x)$ around $x = 0$. Calculate the first 4 terms of the series, i.e. out to and including the term $x^3$.

**Answer:**

$$y' = y + 2y^2$$
$$y'' = y' + 4yy'$$
$$y''' = y'' + 4y'y' + 4yy''$$

Thus $y(0) = 1, y'(0) = 1 + 2 = 3, y''(0) = 3 + 4(1)(3) = 15, y'''(0) = 15 + 4(9) + 4(1)15 = 111$. Therefore the Taylor series is

$$y(x) = 1 + 3x + \frac{15}{2}x^2 + \frac{37}{2}x^3 + O(x^4) \ .$$

Note: the substitution $u = 1/y$ converts the equation to $u' = -u - 2$, giving the solution $y(x) = \frac{1}{3e^{-x}-2}$, and this could be expanded to get the series.

# Numerical Methods

12. Given a set of data points $(x_i, y_i)$, with $x_i \neq x_j$ for all $i \neq j$, a polynomial $p(x)$ is to be fitted to the data.

    (a) Define the terms monomial basis and Lagrange basis for the polynomial and data points.

    (b) For the data $(1, 4)$, $(2, 1)$, $(3, -1)$, use the Lagrange basis to calculate the polynomial that fits the data.

    **Answer:** $y = \frac{1}{2}x^2 - \frac{9}{2}x + 8$

13. Use Newton's method to solve the equation $x \tan x = 1$, using a starting estimate of $x_0 = 1.1$.

    (a) Calculate $x_1$ and $x_2$.

    (b) What are the forward and backward errors associated with $x_2$?

    **Answer:** $x_1 = 0.941167, x_2 = 0.8697589$. The forward absolute error is 0.009 and the backward error is: $x \tan x = 1.03$.

14. Given the set of equations

$$\begin{aligned} 2x + 3y &= 5 \ , \\ x - y &= 3 \ , \\ 3x + 2y &= 6 \ , \end{aligned}$$

find an approximate solution using the theory of least squares.

**Answer:** $x = 34/15, y = -1/15$.

15. Consider the function

$$f(x) = \frac{1 + 2x^2}{1 - x} \ .$$

   (a) Compute the first three terms of the Taylor series for $f(x)$ around $x = 0$.

   (b) Compute the remainder term $R_2(\xi, x)$.

   (c) For the case $x = 1/2$, obtain $\xi$ explicitly.

   You may leave fractional powers symbolic. For example, a root such as $2^{1/4}$ can be left unevaluated.

**Answer:** (a) $1 + x + 3x^2 + R_2(\xi, x)$ . (b) $R_2(\xi, x) = (1/3!)f^{(3)}(\xi)(x)^3 = \frac{3x^3}{(1-\xi)^4}$. (c) $R_2(\xi, 1/2) = f(1/2) - 1 - 1/2 - 3/4 = 3/4$. Therefore $1 - \xi = 2^{-1/4} \approx 0.16$