THE UNIVERSITY OF WESTERN ONTARIO
London Ontario

Applied Mathematics Ph.D. Comprehensive Examination

Part I: 9:00 am - 12:00 pm, 1 May 2024

Instructions:

- The comprehensive exam consists of two parts. This is Part I. Part I consists of mandatory problems and covers basic material. **In Part I, 80% is required for a passing grade.**
- You may use a calculator, pen, and pencil. NO other aids are allowed. Your calculator must NOT be capable of wireless communication or capable of storing and displaying large text files.

Do ALL of the questions in the following four sections

1. **Calculus**

   (a) (i) Show that $g(x) = \frac{x}{x-1}$ is one-to-one. (ii) Assume a function $f(x)$ satisfies $f\left(\frac{x}{x-1}\right) = \frac{3x-1}{3x+1}$, find the explicit formula for $f(x)$.

   (b) Let $f(x) = (\ln x)^2$ and $g(x) = \frac{1}{x^\alpha}$ for $x > 0$, where α is *any positive real number*. Which of these two functions tends to ∞ faster when $x \to 0^+$? Show your work to support your conclusion.

   (c) Let $b(x) = px^2e^{-x}$ where p is a positive parameter. Find the range for p within which $b(x) < x$ for all $x \in (0, \infty)$ (hence, $b(x)$ has no positive fixed point).

   (d) The improper integral $I = \int_0^\infty e^{-x^2}dx$ is widely seen and used. Do the following for I.

   (i) Explain why it is convergent.

   (ii) Find the value of I by relating I^2 to the limit of a double integral in $\mathbb{R}^2$. [Hint: use symmetry of integrals and polar coordinates.]

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2. **Linear Algebra**

   (a) Consider the linear system $AX = B$ where $A = (a_{i,j})_{m\times n}$ is an $m \times n$ real matrix, $B = (b_1, b_2, \cdots, b_m)^T \in \mathbb{R}^m$ is a real vector and $X = (x_1, x_2, \cdots, x_n)^T$ is the unknown vector. Note that A can also be expressed in terms of column vectors: $A = [A_1, A_2, \cdots, A_n]$ where $A_j = (a_{1j}, a_{2j}, \cdots, a_{nk})^T$ for $j = 1, 2, \cdots n$.

   (i) There are several equivalent necessary and sufficient conditions on A and B for $AX = B$ to be compatible (i.e., having solution), state one of them.

   (ii) State a necessary and sufficient condition on A under which, $AX = B$ is compatible for every $B \in \mathbb{R}^m$.

   (iii) When $m = n$, state an alternative condition for each of the above two questions.

   (iv) Explain why $AX = B$ cannot have precisely two (or three or four) solutions.

   (b) Let $v_1 = (1, -1, 1)$ and $v_2 = (1, 1, -1)$ and W be the subspace in $\mathbb{R}^3$ spanned by v_1 and v_2.

   (a) Find the orthogonal projection of $v = (1, 1, 1)$ in W.

   (b) Find the distance of v to W.

(c) Consider the polynomials $x - 1,\ x^2 - 2$ and $x^3 - 3$.

(i) Determine whether these three polynomials are linearly dependent or linearly independent. Show your work.

(ii) Do these three vectors span $P^{(3)}$ (the vector space of all polynomials with degree no more than 3)? Justify your answer.

(d) Find a basis and the dimension of the kernel space Ker$A$ of the matrix

$$A = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 3 & 0 & 3 & -3 \end{pmatrix}.$$

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3. **Numerical Methods**

(a) Consider the system of equations $f(u, v) = v^4 - u^4 = 0,\ g(u, v) = u^2 + v^2 - 1 = 0$. Find the Jacobian Matrix of this system and set up the main iteration step of Newton's method. Starting from the initial point $(u, v) = (-1, +1)$ use 2 Newton iterations to find an approximate solution of the system. Check the accuracy of your solution.

(b) The one-dimensional version of Newton's Method for solving $F(x) = 0$ for differentiable $F(x)$ locally converges quadratically to roots where $F'(x) \neq 0$. Briefly without calculations state the analygous result for differentialble systems such as the one in (a).

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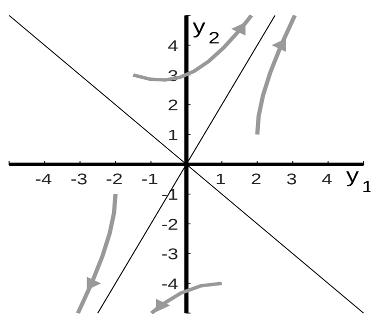
4. **Ordinary differential equations**

(a) Find the general solution to: $y' - 2y = e^{-2x} y^2$.

(b) Give a particular solution to $y'' - y' = 10 \cos 2x$.

(c) Four solution trajectories to $\vec{y}\,' = A\vec{y}$, where A is a real 2×2 matrix, are illustrated in grey on the (y_1, y_2) plane. Give possible eigenvalues and eigenvectors of A (many correct answers are possible). Using these, write the general solution.



(d) i. Give the general solution to

$$2xy - y\sin(xy) \ + \ (\,x^2 - x\sin(xy)\,)\frac{dy}{dx} = 0, \quad x > 0\,.$$

ii. Find the solution if $y(1) = 0$.

iii. If you plotted the solution from Part ii on the (x, y) plane, what slope would it have at this initial condition?

iv. Does a solution exist for initial condition $x_0 = 1$, $y_0 = \pi/2$? Explain why or why not.