THE UNIVERSITY OF WESTERN ONTARIO
London Ontario

Applied Mathematics Ph.D. Comprehensive Examination

24 May 2024
Part II: 9:00 am - 12:00 pm

Instructions: The comprehensive exam consists of two parts. This is Part II, for which a minimum of 60% is required to pass. Students must answer one of two questions in section 1, one of two questions in section 2, and any two of the four questions in section 3.

You may use a calculator, pen, and pencil. NO other aids are allowed. Your calculator must NOT be capable of wireless communication or capable of storing and displaying large text files.

1 Numerical Analysis

Do all the parts of **one** of the two questions below.

NA1. Consider the reaction diffusion PDE $u_t = \nabla^2 u + f(t,u)$ $(\clubsuit)$ which in polar coordinates is $u_t = u_{rr} + \frac{1}{r}u + \frac{1}{r^2}u_{\theta\theta} + f(t,u)$. Suppose that $(\clubsuit)$ is satisfied in the annulus $1 < r < 2$ and satisfies the initial condition $u(0,r,\theta) = g(r)$ for $1 < r < 2$, and boundary conditions $u_r(t,1,\theta) = 0$, $u_r(t,2,\theta) = 0$ for $t \geq 0, 0 \leq \theta \leq 2\pi$.

(a) Briefly why is this IBVP symmetric under rotations in θ? Why does it follow that $u = v(t,r)$? and $v_t = v_{rr} + \frac{1}{r}v + f(t,v)$, $v(r,0) = g(r)$ for $1 < r < 2$ and $v_r(t,1) = 0, v_r(t,2) = 0$ for $t \geq 0$?

(b) Set up a typical cell for an explicit forward finite difference discretization of this IBVP. Express $v(t + \Delta t, r)$ approximately in terms of neighboring approximate grid values of v.

(c) Discuss how boundary and initial conditions would be accommodated.

(d) How would you test your code?

(e) What potential disadvantages are there of the given finite difference scheme? How might the numerical approach be improved?

NA2. (a) Define the terms *Forward error*, *Backward error* and *Condition number*.

(b) The equation $x^4 - 3x + 1$ has 2 real solutions in the interval $[0, 2]$. Approximate the 2 roots using 3 steps of a Newton iteration, starting from an initial estimate of your choosing. Estimate the forward and backward errors in your solutions.

(c) For a square matrix A, the undergraduate way to calculate eigenvalues and eigenvectors is to solve $\det(A - \lambda I) = 0$ and hence solve $(A - \lambda I)x = 0$.

i. Explain how the power method for eigenvalue computation works.

ii. For the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix},$$

use the seed vector $v_0 =< 1, 1, 1 >$ to conduct two steps of the power method, and hence estimate one eigenvalue of the matrix.

iii. Estimate the errors in your approximation.

2 Partial Differential Equations

Do all the parts of **one** of the two questions below.

PDE1. Consider a density $u(x,t) \geq 0$ mass per unit length at position $x \in \mathbb{R}$ and time $t \geq 0$, with $u(x,0) = u_0(x)$ is a given function of $x \in \mathbb{R}$. Suppose this is modeled by the differential conservation law $u_t + \phi_x = f(x,t)$.

(a) For $\phi(x,t) = cu$, briefly interpret u, ϕ, $f(x,t)$ for a possible application. Set up, but do not solve, the characteristic differential equations for the solution of the IVP in this case.

(b) Now suppose that $\phi = -Du_x + cu$. As in (a) give an interpretation for a possible application.

(c) Find the Fourier Transform of $u_t - Du_{xx} = f(x,t)$, $u(x,0) = u_0(x)$ and solve the resulting ODE for the Fourier transform of u in the transform space. See the attached short Table of Fourier Transforms.

(d) Use convolution to simplify the inverse transform to give an integral solution formula for u. Identify a Green's function in your solution and interpret it in terms of the application you described in (b). What famous equation corresponds to the case $D = i$ and $f(x,t) = 0$?

(e) Replacing $f(x,t)$ with $F(u(x,t))$ in the integral solution for u yields a nonlinear integral equation for u. Name one application of this equation.

PDE2. Consider the system of reaction diffusion equations with no-flux boundary condition:

$$\begin{cases} u_t(t,x) = D_1 u_{xx} + g(u,v), \ \ t > 0, \ \ x \in (0,L), \\ v_t(t,x) = D_2 v_{xx} + h(u,v), \ \ t > 0, \ \ x \in (0,L), \\ u_x(t,0) = u_x(t,L) = 0, \ \ v_x(t,0) = v_x(t,L) = 0. \end{cases} \tag{1}$$

(a) For (1), explain what is meant by Turing instability, what is meant by pattern formation, and how these two notions are related.

(b) Can Turing instability occur if $D_1 = D_2$? Explain why.

(c) When $D_1 \neq D_2$, will Turing instability always occur? If not, find conditions on g and f under which pattern formation can be excluded; and conditions under which pattern formation is possible.

(d) Set $D_2 = D, \ D_2 = 1, \ L = \pi, \ \ g(u,v) = 1 - u + u^2 v$ and $h(u,v) = 2 - u^2 v$ in (1), leading to the so called Turing system. Apply your results above to explore the possibility of Turing instability by examining the stability of the unique positive equilibrium $E^+ = (3, 2/9)$.

3 Application Area Questions

Do all the parts of **two** of the four questions below.

AA1. Consider the following discrete time epidemic model. The variables S_t, I_t and R_t correspond to the number of individuals who are susceptible, infectious and recovered (immune) at time t. The total population size is $N_t = S_t + I_t + R_t$. The probability of birth and probability of death are equal and given by parameter $b \in (0, 1)$. The probability of recovery is $\gamma \in (0, 1)$.

$$S_{t+1} = S_t - \frac{\beta}{N_t} I_t S_t + b(I_t + R_t) \tag{2}$$

$$I_{t+1} = I_t(1 - \gamma - b) + \frac{\beta}{N_t} I_t S_t \tag{3}$$

$$R_{t+1} = R_t(1 - b) + \gamma I_t \tag{4}$$

Further assume that $0 < b + \gamma < 1$ and that the infection rate, β is in the interval (0,1).

(a) Demonstrate that N_t is a constant, N, and use this to replace the variable R_t, yielding a system of two difference equations in S_t and I_t only.

(b) Find all equilibria for this system.

(c) Derive the existence and stability conditions for the disease-free equilibrium.

(d) Define $\mathcal{R}_0 = \beta/(\gamma + b)$. Assume $2 - b\mathcal{R}_0 > 0$. Demostrate that the endemic equilibrium exists and is locally asymptotically stable if $\mathcal{R}_0 > 1$. (Hint: recall the difference equation stability condition $|\text{Tr}(J)| < 1 + \det(J) < 2$).

AA2. Assume allele frequency x is x_0 at time 0. Let $p(x|x_0, t)$ be the probability density function, such that the probabilty of finding an allele at a frequency in the interval $(x, x + \delta x)$ in time interval $(t, t + \delta t)$ is given by $p(x|x_0, t)\delta x \delta t$. The backward Kolmogorov equation is given by

$$\frac{\partial p(x|x_0, t)}{\partial t} = \frac{1}{2} V(x_0) \frac{\partial^2 p(x|x_0, t)}{\partial x_0^2} + M(x_0) \frac{\partial p(x|x_0, t)}{\partial x_0} \tag{5}$$

For the Wright-Fisher model of selection and drift, recall that $M(x) = x(1-x)s$ and $V(x) = \frac{x(1-x)}{2N}$, where s is the selection coefficient and N is the population size.

To find the probability of fixation of a selected allele, we let $t \to \infty$ and consider $x = 1$ in the backward Kolmogorov equation. Specifically, in Equation 5 we replace $p(x|x_0, t)$ with $\lim_{t\to\infty} p(x = 1|x_0, t)$ which we can write for convenience as $p(x = 1|x_0, t = \infty)$. (Hint: $\partial p/\partial t = 0$ in this limit).

Define a new function $Q(x_0) = \frac{\partial p(x=1|x_0, t=\infty)}{\partial x_0}$ and use this to solve for the fixation probability, $p(x = 1|x_0, t = \infty)$. Your answer will contain two arbitrary constants. Provide two initial conditions which could be used to determine these constants (but you do not need to determine them explicitly).

AA3. (a) Given two polynomials

$$p(x) = \sum_{k=0}^{n} p_k x^k \ ,$$
$$q(x) = \sum_{k=0}^{m} q_k x^k \ ,$$

describe how the *Sylvester matrix* is formed, and how it is used to decide whether $p(x)$ and $q(x)$ have a common root.

(b) For the following polynomials

$$p(x) = 6x^2 + 13x - 5 \ ,$$
$$q(x) = 6x^2 + 17x + 5 \ ,$$

use the Sylvester matrix to decide whether there is a common root. [Obviously, solving the polynomials using the quadratic formula gets no credit.]

AA4. For the matrix

$$A = \begin{pmatrix} x+1 & -1 & -1 \\ x-2 & 3 & 2 \\ -x & x & x \end{pmatrix} \ ,$$

calculate the Hermite Normal Form.

Concise Table of Fourier Transforms

$f(x)$	$\widehat{f}(k)$
1	$\sqrt{2\pi}\,\delta(k)$
$\delta(x)$	$\dfrac{1}{\sqrt{2\pi}}$
$\sigma(x)$	$\sqrt{\dfrac{\pi}{2}}\,\delta(k) - \dfrac{\mathrm{i}}{\sqrt{2\pi}\,k}$
$\operatorname{sign} x$	$-\,\mathrm{i}\,\sqrt{\dfrac{2}{\pi}}\,\dfrac{1}{k}$
$\sigma(x+a) - \sigma(x-a)$	$\sqrt{\dfrac{2}{\pi}}\,\dfrac{\sin a\,k}{k}$
$e^{-\,a\,x}\,\sigma(x)$	$\dfrac{1}{\sqrt{2\pi}\,(a+\mathrm{i}\,k)}$
$e^{a\,x}\,(1-\sigma(x))$	$\dfrac{1}{\sqrt{2\pi}\,(a-\mathrm{i}\,k)}$
$e^{-\,a\,\lvert\,x\,\rvert}$	$\sqrt{\dfrac{2}{\pi}}\,\dfrac{a}{k^2+a^2}$
$e^{-\,a\,x^2}$	$\dfrac{e^{-\,k^2/(4\,a)}}{\sqrt{2\,a}}$
$\tan^{-1} x$	$-\,\mathrm{i}\,\sqrt{\dfrac{\pi}{2}}\,\dfrac{e^{-\,\lvert\,k\,\rvert}}{k}$
$f(c\,x+d)$	$\dfrac{e^{\,\mathrm{i}\,k\,d/c}}{\lvert\,c\,\rvert}\,\widehat{f}\left(\dfrac{k}{c}\right)$
$\overline{f(x)}$	$\overline{\widehat{f}(-k)}$
$\widehat{f}(x)$	$f(-k)$
$f'(x)$	$\mathrm{i}\,k\,\widehat{f}(k)$
$x\,f(x)$	$\mathrm{i}\,\widehat{f}\,'(k)$
$f * g(x)$	$\sqrt{2\pi}\,\widehat{f}(k)\,\widehat{g}(k)$

Note: The parameters a, c, d are real, with $a > 0$ and $c \neq 0$.