THE UNIVERSITY OF WESTERN ONTARIO
London Ontario

Applied Mathematics Ph.D. Comprehensive Examination

6 May 2025
Part I: 1:00 pm - 4:00 pm

Instructions: The comprehensive exam consists of two parts. This is Part I. Part I consists of mandatory problems and covers basic material. In Part I, 80% is required for a passing grade.

You may use a calculator, pen, and pencil. NO other aids are allowed. Your calculator must NOT be capable of wireless communication or capable of storing and displaying large text files.

PART I: Do ALL of the questions in the following three sections.

Linear Algebra

LA1. Let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be the function which rotates a point counter-clockwise around the origin by $90°$. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the function which reflects a point over the x-axis.

(a) Show that T is a linear transformation.

(b) Represent both R and T as matrices
(i.e. Find the matrices A_R and A_T such that $R(p) = A_R p$ and $T(p) = A_T p$).

(c) Represent the linear transformation $R \circ T$ as a matrix.

LA2. Let

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

(a) Find the eigenvalues of A and their algebraic multiplicities

(b) Find the geometric multiplicity of each eigenvalue of A by computing a basis for each eigenspace.

(c) Finish this definition: An $n \times n$ matrix is *diagonalizable* if...

(d) Is A diagonalizable? Justify your answer.

LA3. (Circle the best way to complete this statement)
If A is a 3×2 matrix, then $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions

(a) Always

(b) Sometimes

(c) Never

and justify your answer.

Calculus

(1) Find $\lim_{x \to 0} \left(\frac{1}{n} \sum_{k=1}^{n} k^x \right)^{\frac{1}{x}}$.

(2) Evaluate $\int_0^1 x\,|x-\beta|\,dx$, where β is a real number.

(3) Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{n}{2^n + (-3)^n} x^{2n-1}$.

(4) Suppose that the function $f(x)$ is continuous in $x \in [0,1]$, and $\int_0^1 f(x)dx = \alpha$, find

$$\int_0^1 dx \int_x^1 f(x)f(y)\,dy.$$

Hint: Change the order of integration.

Numerical Methods

NA1 Suppose that you need to calculate $a^{\frac{1}{n}}$ where a is a real positive constant $a > 0$ and n is a positive integer. Describe two different approaches to solve this problem numerically. Include details of the algorithm used (specify the iterations you would implement), define the function the alorithm would be applied to, and describe the convergence properties (Does it converge? How fast?). Comment on the advantages/disadvantages of your two methods.

NA2 Consider the two-point forward difference numerical approximate of the derivative

$$f'(x) \approx \frac{f(x+h) - f(x)}{h} \tag{1}$$

where $h > 0$. When evaluated in finite precision on a computer Eq. (1) has both truncation and machine error. Which is more important and why? What limits the accuracy of Eq. (1)? How might we improve Eq. (1)? Provide mathematical reasoning, not only words.

Ordinary Differential Equations

(a) Find the explicit, general solution to: $y' = \frac{\sin t}{3-4y}$.

(b) Consider $(4 - x^2)y'' + 2y = 0$, and the point $x_0 = 0$.

   i. Find the recurrence relation for the coefficients of the power series solution.

   ii. Find the first three terms of each of two series solutions y_1 and y_2. If a series terminates in less than three terms, write the full series.

(c) Consider $t^2y'' - t(t + 2)y' + (t + 2)y = 0$, with $t > 0$.

   i. Given that $y_1(t) = t$ is one solution, find the general solution.

   ii. Demonstrate that your general solution consists of a fundamental set of solutions to the differential equation.