THE UNIVERSITY OF WESTERN ONTARIO
London Ontario

Applied Mathematics Ph.D. Comprehensive Examination

14 May 2025
Part II: 1:00 pm - 4:00 pm

Instructions: The comprehensive exam consists of two parts. This is Part II, for which a minimum of 60% is required to pass. Answer all questions in all three sections (A,B and C).

You may use a calculator, pen, and pencil. NO other aids are allowed. Your calculator must NOT be capable of wireless communication or capable of storing and displaying large text files. NO other aids are allowed.

(A) Numerical Analysis

1. Consider a system of equations of the form $Ax = b$ where A is $n \times n$ invertible matrix, x and $\hat{x}$ are $n \times 1$ vectors. Here $\hat{x}$ is an approximate solution of $Ax = b$ so that $A\hat{x} = \hat{b}$.

   (a) The forward error for $Ax = b$ with respect to a vector norm $\|\|$ is $\|x - \hat{x}\|$. Define a corresponding relative forward error? Why is relative forward error usually preferable to forward error?

   (b) The Error Magnification Factor for x, $\hat{x}$ above is defined as EMF $:= \frac{\text{relative FE}}{\text{relative BE}}$. Give the definitions of relative FE and relative BE that so that

   $$\text{EMF} := \frac{\text{relative FE}}{\text{relative BE}} = \frac{\|x - \hat{x}\|}{\|A(x - \hat{x})\|} \frac{\|Ax\|}{\|x\|}$$

   Give a 1 sentence interpretation the EMF.

   (c) The condition number of the matrix A is defined as

   $$\text{ConditionNumber}(A) = \max_{x,\hat{x},x-\hat{x} \in \mathbb{R}^n \setminus \{0\}} \frac{\text{relative FE}}{\text{relative BE}}$$

   Show that

   $$\text{ConditionNumber}(A) = \max_{x,\hat{x},x-\hat{x} \in \mathbb{R}^n \setminus \{0\}} \frac{\|A^{-1}A(x - \hat{x})\|}{\|A(x - \hat{x})\|} \frac{\|Ax\|}{\|x\|} = \max_{x,z \in \mathbb{R}^n \setminus \{0\}} \frac{\|A^{-1}z\|}{\|z\|} \frac{\|Ax\|}{\|x\|} = \|A^{-1}\|\|A\|$$

   with appropriate matrix norms in the last step.

   (d) Discuss the importance of such condition numbers in applications and or theory. [max: 20 words, point form OK].

2. Suppose that the displacement $u(x, t)$ of a string, $u(x, t)$ at position x and time t is modeled by $u_{tt} = -2u_t + u_{xx}$. At $t = 0$ the string satisfies $u(x, 0) = x(1 - x)$, $u_t(x, 0) = 0$. It also has the boundary conditions $u(0, t) = 0$, $u(1, t) = 0$. The task is to describe and discuss methods for approximating the solution on the region $0 \leq x \leq 1$, $0 < t \leq 1$.

(a) Approximate the second order derivatives u_{xx}, u_{tt}, u_t by finite differences. Justify the local order of the finite difference approximation of u_{xx}. State the order of local accuracy the finie difference approximations of u_{tt} and u_t.

(b) Set up a typical cell for an explicit forward finite difference discretization of this IBVP. Express $u(x, t + \Delta t)$ approximately in terms of neighboring approximate grid values of u.

(c) Discuss how boundary and initial conditions would be accommodated. Note you are not asked to describe detailed code, just the main ideas, with short fragments of pseudo-code.

(d) How would you test the code you developed to implement this method?

(e) What potential disadvantages are there of the given finite difference scheme? How might the numerical approach be improved? [max: 30 words, point form OK].

(B) Partial Differential Equations

B-1. Consider the following parabolic equation in the given domain:

$$u_t(x,t) = \boxed{a(x,t)u_{xx}(x,t) + b(x,t)u_x(x,t)} + c(x,t)u + f(x,t),\ t \in (0,T),\ x \in (a,b). \tag{1}$$

Do the following

(i) Accurately state the weak maximum principle for (1), including the conditions and conclusion.

(ii) Show that the maximum principle for the following specific equation does not hold.

$$u_t = x^2 u_{xx}(x,t), \quad (x,t) \in (-1,1) \times (0,1/2) \tag{2}$$

(iii) Does your finding in (ii) contradict the conclusion in (i)? Explain why?

B-2. Assume that the population of a species living in the 1-D domain $x \in (0, L)$ is described by the diffusive logistic equation (also referred to as Fisher-KPP equation) with the given homogeneous boundary condition.

$$\begin{cases} u_t(t,x) = Du_{xx}(t,x) + ru(t,x)[1 - u(x,t)], & t > 0, \ \ x \in (0,L), \\ u(t,0) = 0, \ \ u(t,L) = 0. \end{cases} \tag{3}$$

It is obvious that $u \equiv 0$ is an equilibrium (steady state solution) called the extinction equilibrium. Show that there exists a critical value D^* for diffusion rate D such that

(i) when $D < D^*$, $u = 0$ is unstable, meaning that any positive solution near $u = 0$ will be pushing away from $u = 0$ (hence growing), and this implies that the population cannot go to extinction.

(ii) however, when $D > D^*$, then $u = 0$ becomes stable meaning that it will attract all solutions near $u = 0$, and this implies that the population may go to extinction when it follows below certain level.

(iii) Can you offer a biological/ecological explanation about the the results in (i) and (ii)?

(C) Mathematical Biology

MB1 A number of beetles (denoted n_0) is placed in a box. The box contains the resources needed by the beetles. These resources are replaced regularly so that they don't get used up. The box is otherwise sealed, so beetles cannot leave or enter after the initial catch.

The number of beetles $n(t)$ increases rapidly at first, but after many days the increase is much slower and eventually $n(t)$ is approximately constant.

(a) You model this population using the logistic equation:

$$\frac{dn(t)}{dt} = rn(t)\left(1 - \frac{n(t)}{K}\right) \tag{4}$$

Briefly describe what r and K are in the above equation (at most three sentences each). Based on the observed behavior of $n(t)$ for the beetles described above, are r and K positive or negative? How does K compare to n_0?

(b) Find the equilibrium points of Eq. (4) and classify their stability. Explain what stable/unstable equilibrium points represent biologically in the logistic model.

(c) The person caring for the beetles forgets to keep replacing their resources. $n(t)$ starts to decline until it eventually reaches zero. Describe how r and/or K would need to be adjusted to model this new situation. Find the new equilibrium points and classify their stability (if necessary you may include all real n in your analysis).

(d) How would you use your new logistic model with adjusted r and K to compute a predicted time to extinction? Describe the approach you would take precisely, but you do not need to perform the computation to actually make a prediction.

(e) To better handle situations like (c) above, write down a more detailed model that tracks $n(t)$ and also the resource level $R(t)$. Eq. (4) does not need to be part of your answer, you may create an entirely new model. Define any newly introduced parameters. There are many valid answers here, but to receive full marks you must show that your model captures the behaviors described above, in both the growth and extinction scenarios. For example, with appropriate parameter choices your model should be capable of describing the first scenario where resources are replenished, and in this case there should exist a positive stable equilibrium.

MB2 An organism has the following unusual "double or nothing" life cycle. Each timestep, every individual attempts to produce offspring and then dies. The number of offspring each individual produces is either 0 or 2, chosen at random with equal probabilities.

(a) Let $n(t)$ denote the total number of individuals at timestep t, and $f(t)$ be a random variable denoting the number of inviduals that produce offspring going from timestep t to $t+1$. Show that

$$P(f = x) = \frac{n!}{(n-x)!x!}\frac{1}{2^n} \tag{5}$$

where P denotes the probability of an event and $x! = x(x-1)(x-2)\cdots 1$

(b) Show that the expected abundance remains constant i.e.

$$\mathbb{E}[n(t+1)|n(t)] = n(t)$$

You do not need to re-derive well-known properties of Eq. (5) if you use Eq. (5) in your answer.

(c) Let $p(m,t)$ denote the probability that, starting with m individuals at time t, eventually no descendants of these individuals remain. That is, the probability that the group of individuals formed by offspring of this group, and their offspring's offspring etc. eventually goes extinct. Show that the following Kolmogorov backwards equation holds

$$p(1,t-1) = \frac{1}{2}p(2,t) + \frac{1}{2}p(0,t)$$

(d) Express $p(2,t)$ in terms of $p(1,t)$.

(e) Now use the results of (c) and (d) to derive the probability that, starting from a single individual in an infinite environment, this organism can persist indefinitely.

(D) Disease Modelling

DM1 Consider the following 6-parameter disease model, which includes demographics (births and natural death). Assume $S(t)$, $I(t)$ and $R(t)$ are defined as usual, and that all parameters and state variables are positive.

$$\frac{dS}{dt} = \lambda r S^2 - BSI - \delta S \tag{6}$$

$$\frac{dI}{dt} = BSI - \alpha IR - \delta I \tag{7}$$

$$\frac{dR}{dt} = kIR - \delta R \tag{8}$$

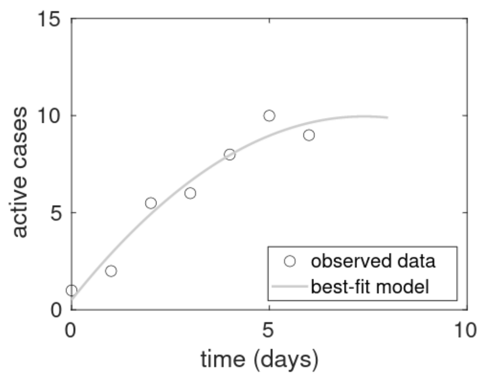
(a) There are several ways in which this system is atypical. Describe in words how births are modelled and give an example of when such a model might be appropriate. Describe in words how recovery from the disease is modelled and give an example (however unrealistic) of disease recovery that might be modelled in this way.

(b) Perform a parameter reduction on this system.
Hint: It's possible to find an equivalent 2-parameter system.

(c) Using the reduced-parameter model, find all the positive equilibria of this system.

(d) Interpret the disease-free equilibrium in terms of the original parameters, and then confirm that this is indeed an equilibrium in the original system.

DM2 A single case of a novel infectious disease is known to have arrived in a region on **Monday**, October 7, "day 0". The numbers of active cases observed in the region each day following day 0 are plotted in the graph below (dots). The regional public health team has fit these data to a model $I = at^2 + bt + c$ where t is time in days, $I(t)$ is the number of active cases on day t, and a, b and c are parameters. The best fit parameters are $a = -0.25$, $b = 3$ and $c = .07$, and the best-fit curve is also plotted by a solid line on the graph.



You may use point form comments, sketches and/or equations for this question.

(a) Comment on this data fitting. You may discuss strengths or weaknesses, the meaning/significance of the best-fit parameters, and why this is a good or a bad approach to fit to the data. You should be able to make 5 or 6 point-form comments here.

(b) The public health team would like to predict when this wave of the epidemic will be over. Is it reasonable to predict this based on this fit?

- If your answer is "yes", explain why. Derive a numerical prediction of when we will have zero cases.
- If your answer is "no", explain why not, and describe an alternative approach for data fitting.

(c) Describe at least one way in which the reliability of the health team's fit to the data could be assessed quantitatively. You may use equations, pseudo-code or sketches, or simply describe in words.