THE UNIVERSITY OF WESTERN ONTARIO
London Ontario

Applied Mathematics Ph.D. Comprehensive Examination

Wednesday June 13, 2018
1:00 - 4:00 pm

The exam consists of two parts. Part I contains mandatory problems and covers basic materials, while Part II covers slightly more advanced materials at graduate course level.

This is **Part II**, for which **60% is required to pass.** You may use a calculator, pen, and pencil. **NO other aids** are allowed. Your calculator **must NOT** be capable of wireless communications or capable of storing and displaying large text files.

Part II. In addition to Question 9 which is mandatory, do 5 out of the Questions 1-8. For Questions 1-8, if you attempt more than 5, only the first 5 to be marked will count.

Dynamical Systems

1. Consider the system
$$\begin{cases} \dot{x} = x - x^3 \\ \dot{y} = -y + \sin(\pi x). \end{cases}$$
Find all equilibria for this system and discuss their stability and geometric types (e.g.,node, saddle, focus, center).

2. Consider the following three systems
$$(A)\ \begin{cases} \dot{x} = y + x(4 - x^2 - y^2)(16 - x^2 - y^2), \\ \dot{y} = -x + y(4 - x^2 - y^2)(16 - x^2 - y^2); \end{cases} \qquad (B)\ \begin{cases} \dot{x} = y + x(4 - x^2 - y^2)^2(16 - x^2 - y^2), \\ \dot{y} = -x + y(4 - x^2 - y^2)^2(16 - x^2 - y^2); \end{cases}$$
$$(C)\ \begin{cases} \dot{x} = y + x(4 - x^2 - y^2)(16 - x^2 - y^2)^2, \\ \dot{y} = -x + y(4 - x^2 - y^2)(16 - x^2 - y^2)^2. \end{cases}$$
For these systems,
   (i) find their limit cycles;
   (ii) determine the stability of the trivial equilibrium (origin), as well as the stability/instability/semi-stability of the limit cycles;
   (iii) sketch their phase portraits.

Numerical Methods

3. Consider the differential equation $y' = ay + b(1 - e^{-t})$ for constant a and b and $a < 0$.
   (a) Find the equilibrium.
   (b) Write down the Backward Euler Method for the equation.

(c) View Backward Euler as a fixed point iteration scheme to prove that the method's approximate solution will converge to the equilibrium as $t \to \infty$.

(d) Find the formula for the second order Taylor method for this problem. Would this be a better/worse scheme that the Backward Euler for this problem? Explain.

4. Consider the initial value problem (IVP):

$$\frac{dy}{dt} = f(t, y), \qquad t \geq a, \qquad y(a) = \alpha.$$

(a) Give conditions on $f(t, y)$ that ensure that the IVP is well-posed.

(b) For $f(t, y) = \sqrt{y}$ and $a = 0,\ \alpha = 0$ the IVP has two solutions. What are they and does this conflict with the result in (a)? Explain why.

(c) By matching terms in a Taylor expansion, derive the local truncation error for the implicit trapezoidal method for a general f,

$$w_{n+1} = w_n + \frac{h}{2}\left[f(t_n, w_n) + f(t_{n+1}, w_{n+1})\right],$$

where w_n is an approximation to $y_n = y(nh)$. Make sure you define what you mean by the truncation error as there are different conventions in place.

(d) Discuss the consistency, stability, and convergence of the implicit trapezoidal method and what would happen if you were to apply it to the $f(t, y)$ in (b).

Partial Differential Equations

5. Assume we have a conserved variable $c(x, t)$. Associate this with a current (hint: you will need divergence). Assume further that the current is proportional to the gradient of $\frac{\delta \mathcal{F}}{\delta c}$ where $\mathcal{F}$ is the Landau *free energy functional* (in physical terms this is the gradient of the chemical potential, but the physical meaning is not needed here to solve this problem) given as

$$\mathcal{F}[c] = \int dx |\nabla c(x, t)|^2 + \frac{a}{2} c^2(x, t) + \frac{b}{4} c^4(x, t)$$

Find the equation of motion (PDE) for c, that is, find the explicit expression for $\frac{\partial c}{\partial t}$

6. A Green's function $G(x, x')$ cannot be discontinous but can be non-smooth. Such a feature has some very important practical consequences, namely, the jump and the continuity conditions. Assume

$$\mathcal{L} = -\left[\frac{d^2}{dx^2} + q(x)\frac{d}{dx} + r(x)\right]$$

and

$$\mathcal{L}G(x, x') = \delta(x - x')$$

where $\delta(x - x')$ is the Dirac delta function. Derive the jump condition.

Mathematical Biology

7. In population genetics, Hardy-Weinberg Law is a basic law governing the frequencies of an allele between two consecutive generations.

   (a) State this law, including the underlying assumptions.

   (b) Try to remove one of those underlying conditions and obtain a difference equation governing the frequencies of an allele between two consecutive generations.

8. Assume that the population of a species is genetically uniform and its growth is governed by the logistic equations

$$x_1' = r_1 x_1 \left(1 - \frac{x_1}{K_1}\right). \tag{1}$$

   It is known that mutation is very common for many biological species. Assume there is a new strain mutated from the original wild strain x_1. Denote the population of this new strain by x_2, and let the intrinsic grow rate and carrying capacity for this mutant strain be r_2 and K_2 respectively. Considering the competition between the wild strain and the mutant strain, we then obtain the following system of ordinary differential equations:

$$\begin{cases} x_1' = r_1 x_1 \left(1 - \dfrac{x_1 + x_2}{K_1}\right) \\ x_2' = r_2 x_2 \left(1 - \dfrac{x_1 + x_2}{K_2}\right). \end{cases} \tag{2}$$

   Very naturally, one would like to ask if the mutant strain will invade the wild one. This question can be translated into the mathematical question: under what condition(s) on the parameters r_1, r_2, K_1 and K_2, all positive solutions will tend to the boundary equilibrium $(0, K_2)$? Find such condition(s) and give your biological explanation on the condition(s).

Commenting On Departmental Colloquia

9. Below is a list of the titles of the departmental colloquium talks happened during September 2017 and April 2018. Choose 4-6 of them that you attended to comment intelligently (i.e. something not in the abstract and enough to convince us that you did attend and benefit), in a few sentences. Be concise and precise please.

   (a) Evolution of sex-specific pathogen virulence

   (b) Dynamic pathologies in pulsatile blood flow

   (c) Outbreak detection with Markov-modulated Poisson processes

   (d) Epidemic dynamics of cholera in non-homogeneous environments

   (e) Risk of Tick-borne Encephalitis transmission in Hungary

   (f) Hidden approximate symmetry

   (g) Data and physics in nearby galaxies

   (h) Evolutionary dynamics in finite populations; a statistical physical approach

   (i) Adventures in electricity finance

   (j) Evolutionary dynamics of proviral DNA

(k) Multi-scale modelling of patho-physiological mechanisms in the human heart and kidney

(l) Modeling Complexity in the Microcirculation: Approaches at Multiple Scales

(m) I can't get no satisfaction:the concepts we need to reconstruct arguments in applied math

(n) Synthetic biology approaches to suppression of antibiotic resistance: toward model-based design

(o) Soft, smart multi-responsive materials under alcoholic intoxication: What can we learn from computer simulation?

(p) The Foundations of Computational Complexity in the Light of Quantum Computing

(q) On an explicit analytical method for solving a class of operator equations

(r) The High School Math Curriculum: why I think it's broke and what is my fix

(s) A malaria transmission model with temperature-dependent incubation period

(t) The Bohemian Eigenvalue Project