THE UNIVERSITY OF WESTERN ONTARIO
London Ontario

Applied Mathematics Ph.D. Comprehensive Examination

27 May 2016
Part II: 1:00 pm - 3:30 pm

Instructions: Part II covers slightly more advanced material and you have the choice of 4 of 7 questions. In Part II, 60% is required for a passing grade.

You may use a calculator, pen, and pencil. NO other aids are allowed. Your calculator must NOT be capable of wireless communication or capable of storing and displaying large text files.

PART II: Do 4 of the following 7 questions. If you attempt more than 4, only the first four to be marked will count.

1. **Partial Differential Equations**

Consider the integral conservation laws for $u(x,t) \geq 0$ and $v(x,t) \geq 0$:

$$\begin{aligned} \frac{\partial}{\partial t}\int_a^b u\,dx &= -\left[cu\right]_a^b + \int_a^b (q\,u - r\,uv)dx \\ \frac{\partial}{\partial t}\int_a^b v\,dx &= -\left[cv\right]_a^b + \int_a^b (-s\,v + r\,uv)dx \end{aligned}$$

Here a and b are arbitrary and q, r, s are strictly positive constants.

(a) Show that the differential form of the above conservation laws is

$$\begin{aligned} u_t + (cu)_x &= q\,u - r\,uv \\ v_t + (cv)_x &= -s\,v + r\,uv \end{aligned}$$

(b) Interpret the terms in (a) with respect to an application. Give and justify one improvement of the model.

(c) The matrix form of the system in (a) is $\begin{pmatrix} u \\ v \end{pmatrix}_t + \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}_x = \begin{pmatrix} q\,u - r\,uv \\ -s\,u + r\,uv \end{pmatrix}$.

Find its characteristics. Is the system strictly hyperbolic?

(d) Use characteristics to set up an equivalent ODE system for (c) subject to smooth initial conditions $u(x,0) = u_0(x)$, $v(x,0) = v_0(x)$ for $x \in \mathbb{R}$. Can shocks form for this system?

2. **Numerical Methods** Consider the initial value problem (IVP):

$$\frac{dy}{dt} = f(t, y), \qquad t \geq a, \qquad y(a) = \alpha$$

(a) Define what is meant by local truncation error for a numerical scheme to solve the IVP. How is this different/similar to the global error?

(b) By matching terms in a Taylor expansion, find the *order* of the local truncation error for: (i)The midpoint method,

$$w_{n+1} = w_n + hf\left(t_n + \frac{h}{2}, w_n + \frac{h}{2}f(t_n, w_n)\right);$$

(ii) Heun's method,

$$w_{n+1} = w_n + \frac{h}{4}\left[f(t_n, w_n) + 3f\left(t_n + \frac{2h}{3}, w_n + \frac{2h}{3}f(t_n, w_n)\right)\right].$$

In (i) and (ii), w_n is an approximation to $y_n = y(nh)$.

(c) Show that both methods in (a) give the same approximations in the case where $f(t, y) = -y + t + 1$ for any choice of h. Why is this true?

(d) Are these methods numerically stable for any choice of h for the f given in (b)? Why or why not? (A calculation is expected to show the result).

3. **Dynamical Systems**

Consider the nonlinear differential system,
$$\begin{aligned} \frac{dx}{dt} &= -x - y + x\,y \\ \frac{dy}{dt} &= -y + 4\,x^3 - 4\,x^4 \end{aligned}$$

(a) Show that the origin $(0, 0)$ is the unique equilibrium of the system.

(b) Use a linear analysis to determine the stability of the equilibrium.

(c) Apply a Lyapunov function to determine the stability of the equilibrium. Is your conclusion different form that of Part (b)? Why?

(d) Can the system have closed orbits (trajectories)? Give your reason to explain.

(e) Sketch the phase portrait of the system.

4. **Computer Algebra**

Consider a PDE for a density $u(x,t) \geq 0$ and its initial condition:

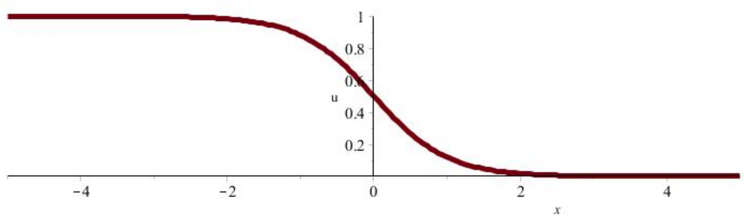
$$u_t = u_{xx} + F(u), \quad u(x,0) = u_0(x), \quad \text{for} \quad x \in \mathbb{R}$$

You can assume without loss that F is analytic with $F(u) > 0$ for $0 < u < 1$ and $F(0) = 0 = F(1)$.

(a) Give one or two word descriptions of possible interpretations of the terms u_{xx}, $F(u)$ in the PDE. Show that the substitution $z = x - ct$, $u = y(z)$ into the PDE above yields the ODE

$$-c\frac{dy}{dz} = \frac{d^2y}{dz^2} + F(y) \qquad (\heartsuit)$$

Suppose that a monotonic traveling wave solution (TWS) with $u(x,0)$ shown in the figure exists for a certain value of $c > 0$. Give a sketch showing how this solution propagates for $t \geq 0$.



(b) Substitute a scaling $y(z) = Y(Z)$, $z = cZ$ into $(\heartsuit)$ and show that the new ODE in the (Z, Y) coordinates is

$$-Y' = \epsilon Y'' + F(Y) \qquad (\diamond)$$

where $\epsilon = \frac{1}{c^2}$ and $Y' = \frac{dY}{dZ}$, $Y'' = \frac{d^2Y}{dZ^2}$.

(c) Assume now that a series solution in ϵ of form $y = Y = Y_0(Z) + Y_1(Z)\epsilon + Y_2(Z)\epsilon^2 + ...$ exists for sufficiently small ϵ for the TWS shown in the figure. Obtain the ODE obeyed by Y_0 and Y_1 resulting from substituting the series into $(\diamond)$ and considering the resulting series in powers of ϵ.

(d) Discuss the use of computer algebra in (c), to determine Y_0, Y_1, and any finite number of terms Y_2, Y_3, ... etc. You should focus on making your procedure as algorithmic as possible from the point of view of computer algebra. You can assume $Y_0(0) = \frac{1}{2}$, $Y_n(0) = 0$ for $n = 1, 2, 3, \cdots$ and should briefly remark why this is appropriate.

5. **Colloquia:** Comment intelligently (i.e. something not in the abstract and enough to convince us that you attended), in a sentence or two, on 10 of the following colloquium topics.

   (a) The Entropy Production Paradox

   (b) Symbolic-numeric algorithms for computing validated results

   (c) A moment matrix approach to symmetric cubatures

   (d) What Is Hybrid Symbolic-Numeric Computation?

   (e) Certain aspects of flows over rough surfaces

   (f) Exact Algorithms for Linear Matrix Inequalities

   (g) Proof of the Wilf-Zeilberger Conjecture

   (h) From physical principles to quantum theory and beyond

   (i) Transmission dynamics and final epidemic size of Ebola Virus Disease outbreaks with varying interventions

   (j) Complex Dynamics in Biological Systems due to Multiple Limit Cycle Bifurcation

   (k) Existence of stationary solutions for some integro-differential equations with anomalous diffusion

   (l) On the Similarity Metric and the Distance Metric

   (m) Seeing Things by Walking on Numbers

   (n) The Life of Pi: A Talk for Pi Day and Other Days

   (o) Sex-specific migration timing in birds

   (p) Creative Telescoping: Theory and Algorithms

   (q) Human Analytics: Understanding Human Behavior in Context

   (r) Large-scale computer simulations of complex membrane models

   (s) Astroinformatics: the big data of the universe

   (t) Cost-effectiveness of using a gene expression profiling test to aid in identifying the primary tumour in patients with cancer of unknown primary

   (u) On Asymptotic Profiles of The Steady States for a Diffusive SIS Epidemic Model with Mass Action Infection Mechanism

   (v) What and Where are Branch Cuts?

   (w) Optimal Trading and Shipping of Agricultural Commodities

6. **Mathematical Biology**

Consider two species X and Y in a competition model:

$$\begin{aligned} \frac{dX}{dt} &= r_x X \left(1 - \frac{X+aY}{K_x}\right) \\ \frac{dY}{dt} &= r_y Y \left(1 - \frac{Y+bX}{K_y}\right) \end{aligned}$$

with all parameters positive.

(a) Non-dimensionalize the above model to yield a simpler, 3-parameter system. Explain in words the meaning of the non-dimensionalized variables (not parameters).

(b) Find all equilibria of the resulting sytem.

(c) Give the existence conditions for the co-existence equilibrium.

(d) Define a Lyapunov function and use it to demonstrate that when the coexistence equilibrium is stable, it is globally asymptotically stable. Hint: $\sum c_i(x_i - x_i^* \log x_i)$.

7. **Quantum Mechanics**

Write down the Hamiltonian, H, for the harmonic oscillator with the mass, m, and frequency, ω, in terms of momentum, p, and space coordinate, x. Define the non-Hermitean operators

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{ip}{m\omega}\right) \qquad a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{ip}{m\omega}\right)$$

which are the creation and the destruction operators respectively.

Evaluate the commutator $[a, a^\dagger]$ and the number operators $a^\dagger a \equiv N$.

If $N|n\rangle = n|n\rangle$, show that $|n\rangle$ is also an eigenvector of H and thus determine the corresponding energy eigenvalues.

Show that

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

and

$$a|n\rangle = \sqrt{n}|n-1\rangle$$

Explain why n has to be a positive integer, then use the above to write out the explicit matrix expectations of the above operators if the eigenvector is simply the unit column vector, with non-zero row n.